DIAS STP 91-05

Thermal Spectra In The Early Universe

Brian P. Dolan

Department of Mathematical Physics
St. Patrick's College
Maynooth, Ireland

and

School of Theoretical Physics
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

ABSTRACT

It is shown that curvature effects, though completely negligible at the present day, can drastically alter thermal spectra at early times in some inflationary cosmological models. This can lead to a modification of the thermal history of the universe before the entropy production. The effects are only present for spatially curved $k \neq 0$ Robertson-Walker models, not for spatially flat $k = 0$ models. If the entropy increase is $>> 10^{87}$ and $k \neq 0$ then the energy density in the quantum fields is much greater, and hence the universe is much younger, at the onset of inflation than would be the case in a spatially flat universe. The effect is not relevant for chaotic inflation.

Inflation is an intriguing and, so far, viable way of avoiding many of the problems associated with the standard model of hot Big Bang cosmology (for a recent comprehensive review see the articles by Linde and by Blau and Guth in [1]). A detailed understanding of the inflationary mechanism however must entail a study of quantum field theory in curved space-times. In particular, for times later than a few $t_{Pl} \sim 5 \times 10^{-44} s$, it is probably a good approximation to consider the gravitational field as classical and attempt a relativistic quantum mechanical treatment of the matter fields in a classical gravitational background. This is a notoriously difficult but nonetheless interesting and important problem, and much work has been done in this area (a standard reference is the book by Birrell and Davies, [2]).

Specifically, it is quantum field theory in a Robertson-Walker background which would be relevant to a study of the hot Big Bang model at early times. As the name implies, the hot Big Bang involves physics at non-zero temperatures. One aspect of this problem therefore is the thermal spectra of quantum fields in this background. For static Robertson-Walker space times, the spectrum for free fields can be calculated exactly ([3]-[5]) and for fields with spin it is *not* Planckian for non-zero spatial curvature. Of course, in the real world the universe is not static and the Robertson-Walker scale factor, $R(t)$, is not a constant so the thermal spectra are not known exactly in general, but as long as the spectrum is only analysed in the frequency range $\nu >> H$, where $H = \dot{R}/R$ is Hubble's "constant", then one would expect the spectrum to be given reasonably accurately by its static form. The purpose of this paper is to point out that, although these distortions are utterly neglible at the present day, if inflation is correct, then they would have been much more important before the era of inflation, for any fields with non-zero spin.

The energy density for a free field, at temperature T, with spin s in an open Einstein ($k = -1$) universe is given by [3] (natural units are used in which $\hbar = c = k_B = 1$, where k_B is Boltzmann's constant),

$$\begin{aligned} \rho_- &= \frac{g_s T^4}{2\pi^2} \int_0^\infty \frac{x^3 + \left(\frac{s}{RT}\right)^2 x}{e^x - (-1)^{2s}} dx \\ &= \frac{g_s \pi^2 T^4}{96} \left[\left(\frac{15 + (-1)^{2s}}{5} \right) + \left(\frac{s}{\pi RT} \right)^2 \left(\frac{9 + 7(-1)^{2s}}{2} \right) \right], \end{aligned} \tag{1}$$

where g_s is the number of degrees of freedom for the field, $g_s = 2$ for photons. $s = 0$ refers only to the conformally coupled wave equation for scalars.

For a closed closed ($k = +1$) Einstein universe it is, ([4][5]),

$$\begin{aligned} \rho_+ &= \frac{g_s}{2\pi^2 R^4} \sum_{n=1}^{\infty} \frac{n^3 - s^2 n}{e^{n/(RT)} - (-1)^{2s}} + \frac{a(s)}{\pi^2 R^4} \\ &= \frac{g_s T^4}{2\pi^2} \frac{1}{RT} \sum_{n=1}^{\infty} \frac{x_n^3 - \left(\frac{s}{RT}\right)^2 x_n}{e^{x_n} - (-1)^{2s}} + \frac{a(s)}{\pi^2 R^4}, \end{aligned} \tag{2}$$

where $x_n = n/(RT)$. The second term here represents the Casimir energy on S^3, [5]. Again, the $s = 0$ case is only for the conformally coupled scalar wave equation. In particular the spin dependent co-efficients are $a(0) = 1/480$, $a(1/2) = 17/1920$, $a(1) = 11/240$.

For a static spatially flat ($k = 0$) universe, there is no deviation from the usual Planck form and the considerations in this paper are irrelevant.

The deviations from the Planckian shape are of order $1/(RT)^2$. For the present day cosmic microwave backround, with $T = 2.7^o\mathrm{K} \sim (1mm)^{-1}$ and $R \sim 10^{28}\mathrm{cm}$, $RT \approx 10^{29}$ and so the distortion from the Planckian spectrum is less than one part in 10^{58} and is completely negligible. This is even smaller than the recently calculated distortion due to fermion one loop effects in finite temperature QED for $T < 511keV$, [6]. These do not affect the shape of the energy spectrum, merely its overall normalisation, which can be absorbed into an effective dielectric constant $\kappa > 1$. The effects are of order $\alpha^2(T/m_e)^4 \sim 10^{-43}(T^oK)^4$, which is less than 10^{-41} for the present microwave backround, many orders of magnitude larger than curvature effects, but still completely negligible.

Without inflation, the distortions are negligible at all earlier times. This follows simply from the adiabatic expansion of the standard Big Bang model. Since the total entropy, S, is constant we have $S = sR^3 \sim T^3R^3 =$ constant, where $s \sim T^3$ is the entropy density. Thus as long as the total entropy remains constant, TR remains constant and the distortions are always negligible - they are negligible precisely because the entropy of the universe is so big. This is really just another statement of the horizon problem - pushed right back to the Planck time, t_{Pl}. To see this let us make the seemingly natural assumption that at $t_{Pl} = 5 \times 10^{-44}s$ the universe started out at the Planck size, with scale factor $R \approx L_{Pl} = 1.6 \times 10^{-35}m$ in thermal equilibrium at the Planck temperature $T_{Pl} = 1.4 \times 10^{32}\ {}^oK$, and evolved according to the radiation dominated Friedmann equation. Then $T \propto 1/\sqrt{t}$ and $R \propto \sqrt{t}$. At the present time,

$$t \approx 3 \times 10^{17} secs \approx 6 \times 10^{60} t_{Pl}, \tag{3}$$

the temperature would be

$$T_{Present} \approx 4 \times 10^{-31} T_{Pl} \approx 60^o K \tag{4}$$

Allowing for the fact that $T \propto 1/t^{2/3}$, rather than $T \propto 1/t^{1/2}$ after the universe becomes matter dominated, at about $100,000\ years$, or 10^{-5} of its present age, brings this down by another factor of ten to 6^oK. † However, the scale factor R increases by the same factor $(1/4) \times 10^{32}$ as the temperature has decreased, i.e. the present scale factor would be

$$R_{Present} \approx (1/4) \times 10^{32} L_{Pl} \approx 4 \times 10^{-4} m \tag{5}$$

(which is, of course, of the order of the wavelength of 6^oK radiation). The present universe has size $R \approx 10^{26}m$, which is 2.5×10^{29} times bigger than this. This is the horizon problem. Thus a "natural" value of RT would be unity, whereas the observed value is $\sim 10^{29}$. This is essentially another way of saying that a "natural" value of the entropy would be one, whereas the observed value is 10^{87}, which is Guth's original viewpoint, [8].

† It is remarkable that this is so close to the present value of 2.7^oK, this has been noted before, [7]. One cannot help feeling that Nature is giving us a clue to quantum gravity here, the microwave background is at just the right temperature to be the relic of a quantum gravity effect.

However if the total entropy of the universe at an earlier epoch was ever smaller than it is today, $1/(RT)$ would have been larger and the distortions from the Planck spectrum correspondingly larger. This is exactly what inflation does. It is designed to hold T fixed while R increases by at least twenty nine orders of magnitude, thus since $RT \approx 10^{29}$ after inflation, we must have $RT \lesssim 1$ before inflation, and the distortion from a Planckian spectrum for fields with non-zero spin is significant, and even dominant if $RT << 1$. Note that these arguments would not apply to chaotic inflation, since in these models inflation sets in just after the Planck time and there would never have been a time at which both $RT \lesssim 1$ and quantum gravity effects could have been ignored.

The shape of the spectra for photons is shown in figures 1 and 2 for open and closed universes respectively, for various values of RT. For $k = -1$ the energy density per unit frequency is non-zero at zero frequency (specifically it is $T^4/(\pi RT)^2$) and for $RT \lesssim 10$ there is much more energy density at low frequencies than in the Planck spectrum, the distortion being larger for smaller values of RT. For values of RT less than 0.619 there is no longer a peak and the spectrum is monotonically decreasing.

For $k = +1$ the thermal spectrum is discrete and there is less energy at low frequencies than in the Planck spectrum. For $RT << 1$, the thermal spectrum is strongly supressed and almost all the energy is in a single frequency. This phenomenon can be easily understood from the fact that three space is compact. Consider a cube of volume L^3 containing thermal radiation at a temperature such that $1/L >> T$. Then it is not possible to fit a photon of wavelength $1/T$ into the box, and the energy density is exponentially supressed.

For values of $RT << 1$ (i.e. inflation factors $>> 10^{29}$) the thermal history of the universe before inflation is also modified. For the Planck spectrum $\rho_0 = \frac{\pi^2}{30} N_{eff} T^4$ where $N_{eff} = N_b + \frac{7}{8} N_f$ is the effective number of degrees of freedom (1 per bosonic degree of freedom and 7/8 per fermionic degree of freedom). But for $k \neq 0$, $\rho_{\pm} >> \rho_0$ when $RT << 1$ (by a factor of order $(RT)^{-2}$ for $k = -1$ and $(RT)^{-4}$ for $k = +1$ - in the latter case, the Casimir energy is the dominant contribution). In the pre-inflation, radiation dominated era the age of the universe goes like $t \propto 1/\sqrt{\rho}$. Thus, for a given temperature, a $k \neq 0$ universe is much younger than a $k = 0$ one would be (by a factor of order RT for $k = -1$ and $(RT)^2$ for $k = +1$). Thus, in contrast to the usual assumption, the thermal history of the very early, pre-inflation, universe is very sensitive to the value of k, if the inflation factor is $>> 10^{29}$.

As noted above, these spectra cannot be trusted for frequencies $\lesssim H$. In the radiation dominated era, before inflation, $H = 1/2t$ thus the spectra are only to be trusted for

$$x = \frac{\nu}{T} >> \frac{1}{2tT}. \tag{6}$$

The temperature can be eliminated using the Friedmann equation

$$H^2 = \frac{1}{(2t)^2} = \frac{8\pi}{3} G\rho. \tag{7}$$

Thus for example for $k = -1$

$$T^2 = \sqrt{\frac{45}{4\pi^3 G \mathcal{N}(RT)}} \left(\frac{1}{2t}\right) \tag{8}$$

where $\mathcal{N}(RT)$ is obtained from equation (1) (including only fields with $s = 0, 1$, or $1/2$)

$$\mathcal{N}(RT) = N_b \left(1 + \frac{5}{2(\pi RT)^2}\right) + N_f \left(\frac{7}{8} + \frac{5}{64(\pi RT)^2}\right).$$

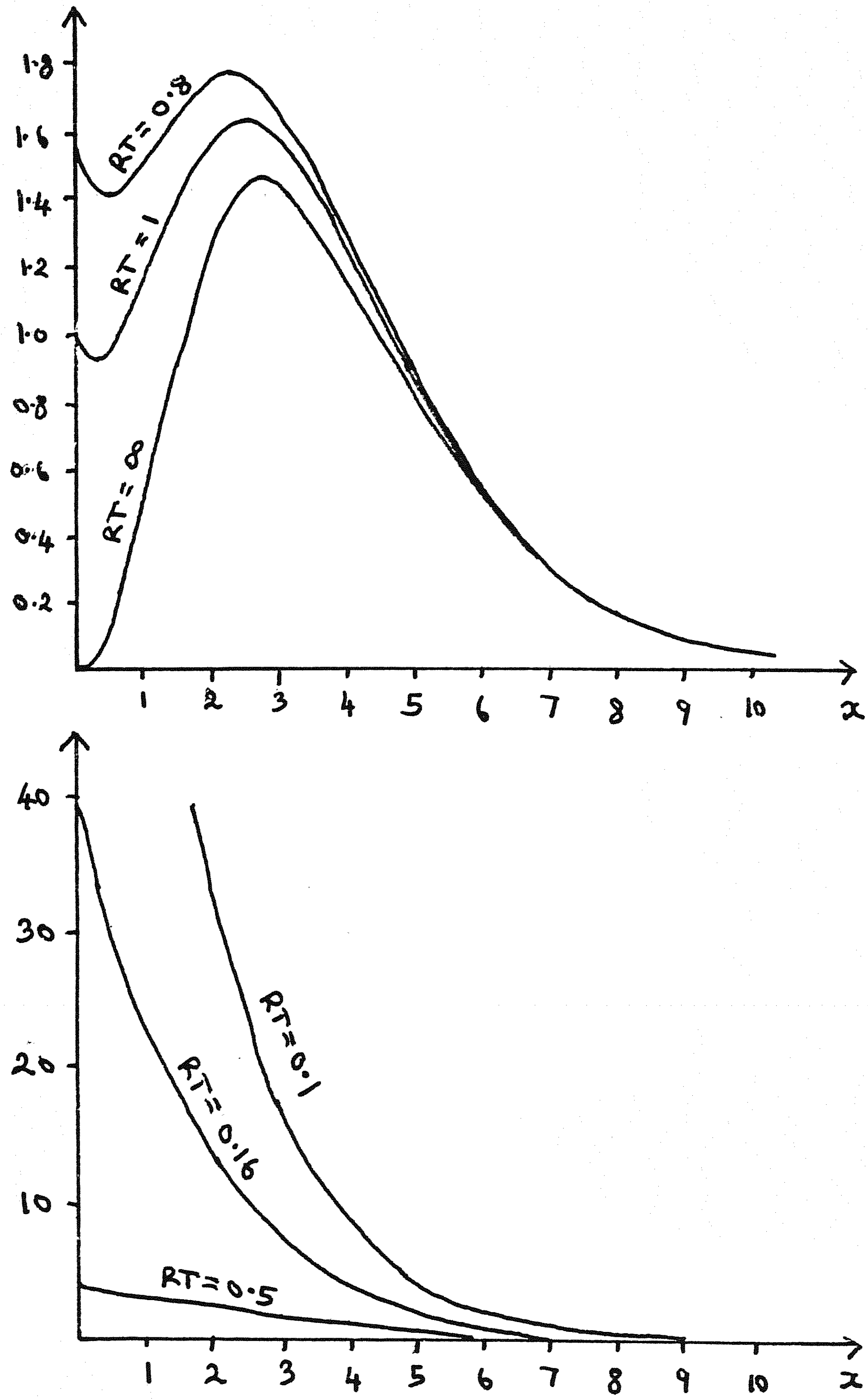
Thus equation (6) gives

$$x >> 1.3 \times 10^{-21} \mathcal{N}(RT)^{1/4} / \sqrt{t(secs)}. \tag{9}$$

If, for example, inflation sets in at $t \sim 10^{-36}s$ with $RT \sim 1$ when the temperature is $\sim 10^{15} GeV$, then just prior to this we must have $x >> 10^{-3}$ for the spectra to be reliable. At earlier times they are less reliable until at the Planck time we would require $x >> 1$, were it not for the additional complication that quantum gravity effects cannot be ignored at this time, so the spectra are completely unreliable at all x. However, between a time of say $10^{-40}s$ and the onset of inflation at $10^{-36}s$ these spectra would be reasonably accurate for $x >> 0.1$.

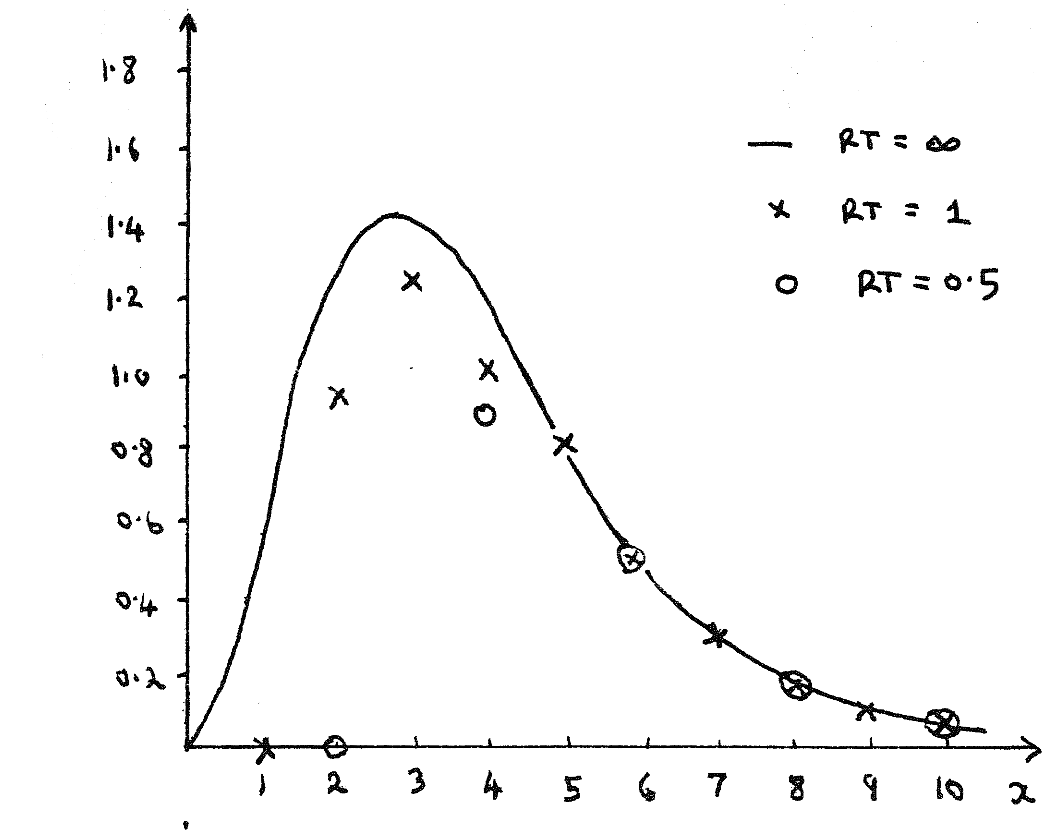
To summarise it has been argued that, while distortions of thermal spectra for fields of non-zero spin due to curvature effects in $k \neq 0$ universes are negligible at the present day, they would have been very significant in any theory with a large production of entropy $\gtrsim 10^{87}$ at any time later than a few Planck times. In particular for new or extended inflationary models these effects would be expected to be important before the era of inflation. For $k \neq 0$ there would have been an earlier onset of inflation than for $k = 0$, if the entropy increase is $>> 10^{87}$. There is no modification of the Planck spectrum for spin zero fields or spatially flat universes.

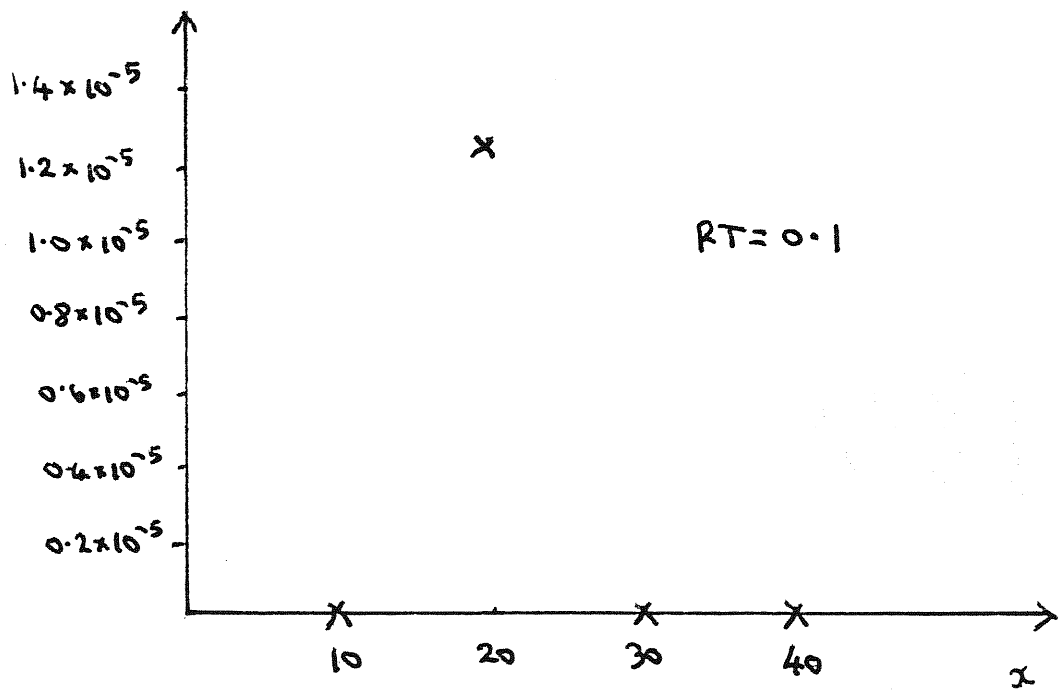
Figure 1. The thermal spectrum for photons in an open (k=-1) static universe for various values of RT, $x = h\nu/k_BT$ is continuous.



$$f(x) = \frac{x^3 + \frac{x}{(RT)^2}}{(e^x - 1)}$$

Figure 2. The thermal spectrum for photons in a closed (k=+1) static universe for various values of RT, $x = n/RT$, $n = 1, 2, \ldots$ is a discrete variable.





$$f(x_n) = \frac{1}{(RT)} \cdot \frac{x_n^3 - \frac{x_n}{(RT)^2}}{(e^{x_n} - 1)}$$

REFERENCES

[1] *300 Years of Gravitation*
ed. S.W. Hawking and W. Israel 1987 C.U.P.

[2] *Quantun Fields In Curved Space*
N.B. Birrell and P.C.W. Davies 1982 C.U.P.

[3] P. Candelas and J.S Dowker 1979 Phys. Rev. D **19** 2902

[4] B. Mashhoon 1973 Phys. Rev. D **8** 4279

[5] L.H. Ford 1976 Phys. Rev. D **14** 3304

[6] G. Barton 1991 Ann. Phys. **205** 49

[7] L. Parker 1976 Nature (London) **261** 20

[8] A.H. Guth 1981 Phys. Rev. D **23** 347